

Flips in Graphs

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Abstract

We study a problem motivated by a question related to quantum-error-correcting codes. Combinatorially, it involves the following graph parameter:

$$f(G) = \min \{ |A| + |\{x \in V \setminus A : d_A(x) \text{ is odd}\}| : A \neq \emptyset \},$$

where V is the vertex set of G and $d_A(x)$ is the number of neighbors of x in A . We give asymptotically tight estimates of f for the random graph $G_{n,p}$ when p is constant. Also, if

$$f(n) = \max \{ f(G) : |V(G)| = n \}$$

then we show that $f(n) \leq (0.382 + o(1))n$.

1 Introduction

In this paper we consider a problem which is motivated by a question from quantum-error-correcting codes. To see how to use graphs to construct quantum-error-correcting codes see, *e.g.*, [2, 4, 5].

Given a graph G with ± 1 signs on vertices, each vertex can perform at most one of the following three operations: O_1 (flip all of its neighbors, *i.e.*, change their signs), O_2 (flip itself), and O_3 (flip itself and all of its neighbors). We want to start with all $+1$'s, execute some non-zero number of operations and return to all $+1$'s. The *diagonal distance* $f(G)$ is the minimum number of operations needed (with each vertex doing at most one operation).

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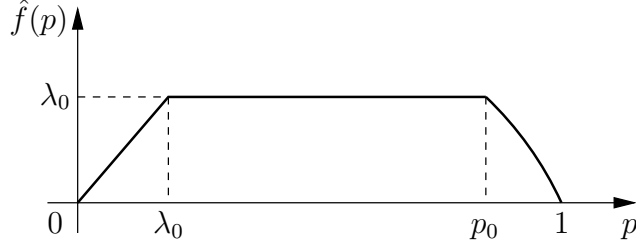


Figure 1: The behavior of $\hat{f}(p) = \lim_{n \rightarrow \infty} f(G_{n,p})/n$ as a function of p .

Trivially,

$$f(G) \leq \delta(G) + 1 \quad (1)$$

holds, where $\delta(G)$ denotes the minimum degree. Indeed, a vertex with the minimum degree applies O_1 and then its neighbors fix themselves applying O_2 . Let

$$f(n) = \max f(G),$$

where the maximum is taken over all non-empty graphs of order n . Shiang Yong Looi (personal communication) asked for a good approximation on $f(n)$.

In this paper we asymptotically determine the diagonal distance of the random graph $G_{n,p}$ for any $p \in (0, 1)$.

We denote the *symmetric difference* of two sets A and B by $A \triangle B$ and the *logarithmic function* with base e as \log .

Theorem 1.1 *There are absolute constants $\lambda_0 \approx 0.189$ and $p_0 \approx 0.894$, see (6) and (12), such that for $G = G_{n,p}$ asymptotically almost surely:*

- (i) $f(G) = \delta(G) + 1$ for $0 < p < \lambda_0$ or $p = o(1)$,
- (ii) $|f(G) - \lambda_0 n| = \tilde{O}(n^{1/2})$ for $\lambda_0 \leq p \leq p_0$,
- (iii) $f(G) = 2 + \min_{x,y \in V(G)} |(N(x) \triangle N(y)) \setminus \{x, y\}|$ for $p_0 < p < 1$ or $p = 1 - o(1)$.

(Here $\tilde{O}(n^{1/2})$ hides a polylog factor).

Figure 1 visualizes the behavior of the diagonal distance of $G_{n,p}$. In addition to Theorem 1.1 we find the following upper bound on $f(n)$.

Theorem 1.2 $f(n) \leq (0.382 + o(1))n$.

In the remainder of the paper we will use a more convenient restatement of $f(G)$. Observe that the order of execution of operations does not affect the final outcome. For any $A \subset V = V(G)$, let B consist of those vertices in $V \setminus A$ that have odd number of neighbors in A . Let $a = |A|$ and $b = |B|$. We want to minimize $a + b$ over all non-empty $A \subset V(G)$. The vertices of A do an O_1/O_3 operation, depending on the even/odd parity of their neighborhood in A . The vertices in B then do an O_2 -operation to change back to $+1$.

2 Random Graphs for $p = 1/2$

Here we prove a special case of Theorem 1.1 when $p = 1/2$. This case is somewhat easier to handle.

Let $G = G_{n,1/2}$ be a binomial random graph. First we find a lower bound on $f(G)$. If we choose a non-empty $A \subset V$ and then generate G , then the distribution of b is binomial with parameters $n - a$ and $1/2$, which we denote here by $\text{Bin}(n - a, 1/2)$. Hence, if l is such that

$$\sum_{a=1}^{l-1} \binom{n}{a} \Pr(\text{Bin}(n - a, 1/2) \leq l - 1 - a) = o(1), \quad (2)$$

then asymptotically almost surely the diagonal distance of G is at least l .

Let $\lambda = l/n$ and $\alpha = a/n$. We can approximate the summand in (2) by

$$2^{n(H(\alpha) + (1-\alpha)(H(\frac{\lambda-\alpha}{1-\alpha}) - 1) + O(\log n/n))}, \quad (3)$$

where H is the binary entropy function defined as $H(p) = -p \log_2 p - (1-p) \log_2 (1-p)$. For more information about the entropy function and its properties see, *e.g.*, [1]. Let

$$g_\lambda(\alpha) = H(\alpha) + (1-\alpha) \left(H\left(\frac{\lambda-\alpha}{1-\alpha}\right) - 1 \right). \quad (4)$$

The maximum of $g_\lambda(\alpha)$ is attained exactly for $\alpha = 2\lambda/3$, since

$$g'_\lambda(\alpha) = \log_2 \frac{2(\lambda - \alpha)}{\alpha}.$$

Now the function

$$h(\lambda) = g_\lambda(2\lambda/3) \quad (5)$$

is concave on $\lambda \in [0, 1]$ since

$$h''(\lambda) = \frac{1}{(\lambda - 1)\lambda \log 2} < 0.$$

Moreover, observe that $h(0) = -1$ and $h(1) = H(2/3) - 1/3 > 0$. Thus the equation $h(\lambda) = 0$ has a unique solution λ_0 and one can compute that

$$\lambda_0 = 0.1892896249152306 \dots \quad (6)$$

Therefore, if $\lambda = \lambda_0 - K \log n/n$ for large enough $K > 0$, then the left hand side of (2) goes to zero and similarly for $\lambda = \lambda_0 + K \log n/n$ it goes to infinity. In particular, $f(G) > (\lambda_0 - o(1))n$ asymptotically almost surely.

Let us show that this constant λ_0 is best possible, *i.e.*, asymptotically almost surely $f(G) < (\lambda_0 + K \log n/n)n$. Let $\lambda = \lambda_0 + K \log n/n$, n be large, and $l = \lambda n$. Let $\alpha = 2\lambda/3$ and $a = \lfloor \alpha n \rfloor$. We pick a random a -set $A \subset V$ and compute b . Let X_A be an indicator random variable so that $X_A = 1$ if and only if $b = b(A) \leq l - a$. Let $X = \sum_{|A|=a} X_A$. We succeed if $X > 0$.

The expectation $E(X) = \binom{n}{a} \Pr(Bin(n - a, 1/2) \leq l - a)$ tends to infinity, by our choice of λ . We now show that $X > 0$ asymptotically almost surely by using the Chebyshev inequality. First note that for $A \cap C \neq \emptyset$ we have

$$Cov(X_A, X_C) = \Pr(X_A = X_C = 1) - \Pr(X_A = 1)\Pr(X_C = 1) = 0.$$

Indeed, if $x \in V \setminus (A \cup C)$, then $\Pr(x \in B(A) | X_C = 1) = 1/2$, since $A \setminus C \neq \emptyset$ and no adjacency between x and all vertices in $A \setminus C$ is exposed by the event $X_C = 1$. Similarly, if $x \in C \setminus A$, then $A \cap C \neq \emptyset$ and an adjacency between x and $A \cap C$ is independent of the occurrence of $X_C = 1$. This implies that $\Pr(x \in B(A) | X_C = 1) = 1/2$ as well. Thus $\Pr(X_A = 1 | X_C = 1) = \Pr(Bin(n - a, 1/2) \leq l - a) = \Pr(X_A = 1)$, and consequently, $Cov(X_A, X_C) = 0$.

Now consider the case when $A \cap C = \emptyset$. Let s be a vertex in A . Define a new indicator random variable Y which takes the value 1 if and only if $|B(C) \setminus \{s\}| \leq l - a$. Observe that

$$\begin{aligned} \Pr(Y = 1) &= \Pr(Bin(n - a - 1, 1/2) \leq l - a) \\ &\leq 2 \Pr(Bin(n - a, 1/2) \leq l - a) = 2 \Pr(X_A = 1). \end{aligned}$$

Moreover,

$$\Pr(X_A = 1 | Y = 1) = \Pr(Bin(n - a, 1/2) \leq l - a) = \Pr(X_A = 1),$$

since for every $x \in V \setminus A$ the adjacency between x and s is not influenced by $Y = 1$. Finally note that $X_C \leq Y$. Thus,

$$\begin{aligned} Cov(X_A, X_C) &\leq \Pr(X_A = X_C = 1) \leq \Pr(X_A = Y = 1) \\ &= \Pr(Y = 1) \Pr(X_A = 1 | Y = 1) \leq 2 (\Pr(X_A = 1))^2. \end{aligned}$$

Consequently,

$$\begin{aligned} Var(X) &= E(X) + \sum_{A \cap C \neq \emptyset, A \neq C} Cov(X_A, X_C) + \sum_{A \cap C = \emptyset} Cov(X_A, X_C) \\ &\leq E(X) + 2 \sum_{A \cap C = \emptyset} (\Pr(X_A = 1))^2 \\ &= E(X) + 2 \binom{n}{a} \binom{n-a}{a} (\Pr(X_A = 1))^2 = o(E(X)^2), \end{aligned}$$

as $E(X) = \binom{n}{a} \Pr(X_A = 1)$ tends to infinity and $\binom{n-a}{a} = o(\binom{n}{a})$. Hence, Chebyshev's inequality yields that $X > 0$ asymptotically almost surely.

Remark 2.1 A version of the well-known Gilbert-Varshamov bound (see, *e.g.*, [3]) states that if

$$2^{-n} \sum_{i=1}^{l-1} \binom{n}{i} 3^i < 1, \quad (7)$$

then $f(n) \geq l$. Observe that this is consistent with bound (2). Let $\lambda = l/n$. We can approximate the left hand side of (7) by

$$2^{n(H(\lambda) + \lambda \log_2 3 - 1 + o(1))}.$$

One can check after some computation that

$$H(\lambda) + \lambda \log_2 3 - 1 = g_\lambda(2\lambda/3).$$

Therefore, (2) and (7) give asymptotically the same lower bound on $f(n)$.

3 Random Graphs for Arbitrary p

Let $G = G_{n,p}$ be a random graph with constant $p \in (0, 1)$.

Observe that for a fixed set $A \subset V$, $|A| = a$, the probability that a vertex from $V \setminus A$ belongs to $B(A)$ is

$$p(a) = \sum_{0 \leq i < \frac{a}{2}} \binom{a}{2i+1} p^{2i+1} (1-p)^{a-(2i+1)} = \frac{1 - (1-2p)^a}{2}.$$

(If this is unfamiliar, expand $(1-2p)^n$ as $((1-p) - p)^n$ and compare).

3.1 $0 < p < \lambda_0$

For $p < \lambda_0$ we begin with the upper bound $f(G) \leq \delta(G) + 1$, see (1). For the lower bound it is enough to show that

$$\sum_{2 \leq a \leq pn} \binom{n}{a} \Pr(Bin(n-a, p(a)) \leq pn-a) = o(1), \quad (8)$$

since $\delta(G) + 1 \leq np$ asymptotically almost surely. (We may assume that $p = \Omega(\frac{\log n}{n})$; for otherwise $\delta(G) = 0$ with high probability and the theorem is trivially true.) This implies that if $|A| + |B| \leq pn$, then $|A| = 1$.

3.1.1 p Constant

We split this sum into two sums for $2 \leq a \leq \sqrt{n}$ and $\sqrt{n} < a \leq pn$, respectively. Let $X = \text{Bin}(n - a, p(a))$ and

$$\varepsilon = 1 - \frac{pn - a}{(n - a)p(a)} \geq 1 - \frac{p}{p(2)} = 1 - \frac{1}{2 - 2p} > 0. \quad (9)$$

Thus, by Chernoff's bound,

$$\Pr(\text{Bin}(N, \rho) \leq (1 - \theta)N\rho) \leq e^{-\theta^2 N\rho/2} \quad (10)$$

we see that

$$\begin{aligned} \Pr(\text{Bin}(n - a, p(a)) \leq pn - a) &= \Pr(X \leq (1 - \varepsilon)E(X)) \\ &\leq \exp\{-\varepsilon^2 E(X)/2\} \\ &= \exp\{-\Theta(n)\}, \end{aligned}$$

and consequently,

$$\begin{aligned} \sum_{2 \leq a < \sqrt{n}} \binom{n}{a} \Pr(\text{Bin}(n - a, p(a)) \leq pn - a) &\leq \sqrt{n} \binom{n}{\sqrt{n}} \exp\{-\Theta(n)\} \\ &\leq \exp\{O(\sqrt{n} \log n)\} \exp\{-\Theta(n)\} \\ &= o(1). \end{aligned}$$

Now we bound the second sum corresponding to $\sqrt{n} < a \leq pn$. Note that

$$\begin{aligned} \sum_{\sqrt{n} \leq a \leq pn} \binom{n}{a} \Pr(\text{Bin}(n - a, p(a)) \leq pn - a) \\ = \sum_{\sqrt{n} \leq a \leq pn} \binom{n}{a} \Pr\left(\text{Bin}\left(n - a, \frac{1}{2} + O(e^{-\Omega(n^{1/2})})\right) \leq pn - a\right) \\ \leq n 2^{nh(p) + o(1)} = o(1). \end{aligned}$$

Here h is defined in (5) and the right hand limit is zero since $p < \lambda_0$.

3.1.2 $p = o(1)$

We follow basically the same strategy as above and show that (8) holds for large a and something similar when a is small. Suppose then that $p = 1/\omega$ where $\omega = \omega(n) \rightarrow \infty$. First consider those a for which $ap \geq 1/\omega^{1/2}$. In this

case $p(a) \geq (1 - e^{-2ap})/2$. Thus,

$$\begin{aligned} \sum_{\substack{ap \geq 1/\omega^{1/2} \\ a \leq np}} \binom{n}{a} \Pr(\text{Bin}(n-a, p(a)) \leq pn-a) \\ = \sum_{\substack{ap \geq 1/\omega^{1/2} \\ a \leq np}} e^{O(n \log \omega / \omega)} e^{-\Omega(n/\omega^{1/2})} = o(1). \end{aligned}$$

If $ap \leq 1/\omega^{1/2}$ then $p(a) = ap(1 + O(ap))$. Then

$$\begin{aligned} \sum_{\substack{ap < 1/\omega^{1/2} \\ 2 \leq a \leq np}} \binom{n}{a} \Pr(\text{Bin}(n-a, p(a)) \leq pn-a) \\ \leq \sum_{\substack{ap < 1/\omega^{1/2} \\ 2 \leq a \leq np}} \left(\frac{ne}{a} e^{-np/10} \right)^a = o(1) \quad (11) \end{aligned}$$

provided $np \geq 11 \log n$.

If $np \leq \log n - \log \log n$ then $G = G_{n,p}$ has isolated vertices asymptotically almost surely and then $f(G) = 1$. So we are left with the case where $\log n - \log \log n \leq np \leq 11 \log n$.

We next observe that if there is a set A for which $2 \leq |A|$ and $|A| + |B(A)| \leq np$ then there is a minimal size such set. Let $H_A = (A, E_A)$ be a graph with vertex set A and an edge $(v, w) \in E_A$ if and only if v, w have a common neighbor in G . H_A must be connected, else A is not minimal. So we can find $t \leq a-1$ vertices T such that $A \cup T$ spans at least $t+a-1$ edges between A and T . Thus we can replace the estimate (11) by

$$\begin{aligned} \sum_{\substack{ap < 1/\omega^{1/2} \\ 2 \leq a \leq np}} \sum_{t=1}^{a-1} \binom{n}{a} \binom{n}{t} \binom{ta}{t+a-1} p^{t+a-1} \Pr(\text{Bin}(n-a-t, p(a)) \leq pn-a) \\ \leq \sum_{\substack{ap < 1/\omega^{1/2} \\ 2 \leq a \leq np}} \sum_{t=1}^{a-1} \left(\frac{ne}{a} \right)^a \left(\frac{ne}{t} \right)^t \left(\frac{taep}{t+a-1} \right)^{t+a-1} e^{-anp/10} \\ \leq \frac{1}{e^2 np} \sum_{\substack{ap < 1/\omega^{1/2} \\ 2 \leq a \leq np}} a ((e^2 np)^2 e^{-np/10})^a = o(1). \end{aligned}$$

3.2 $p_0 < p < 1$

First let us define the constant p_0 . Let

$$p_0 \approx 0.8941512242051071 \dots \quad (12)$$

be a root of $2p - 2p^2 = \lambda_0$. For the upper bound let $A = \{x, y\}$, where x and y satisfy $|N(x) \triangle N(y)| \leq |N(x') \triangle N(y')|$ for any $x', y' \in V(G)$. Then $B = B(A) = N(x) \triangle N(y)$, and thus, asymptotically almost surely $|B| \leq (2p - 2p^2)n$ plus a negligible error term $o(n)$. (We may assume that $1 - p = \Omega(\frac{\log n}{n})$; for otherwise we have two vertices of degree $n - 1$ with high probability, and hence, $f(G)=2$.)

To show the lower bound it is enough to prove that

$$\sum_{3 \leq a \leq (2p-2p^2)n} \binom{n}{a} \Pr(Bin(n-a, p(a)) \leq (2p-2p^2)n-a) = o(1).$$

Indeed, this implies that if $|A| + |B| \leq (2p - 2p^2)n$, then $|A| = 1$ or 2 . But if $|A| = 1$, then in a typical graph $|B| = (p + o(1))n > (2p - 2p^2)n$ since $p > 1/2$.

3.2.1 p Constant

As in the previous section we split the sum into two sums for $3 \leq a \leq \sqrt{n}$ and $\sqrt{n} < a \leq pn$, respectively. Let

$$\varepsilon = 1 - \frac{(2p - 2p^2)n - a}{(n - a)p(a)} \geq 1 - \frac{2p - 2p^2}{p(a)} > 0.$$

To confirm the second inequality we have to consider two cases. The first one is for a odd and at least 3. Here,

$$1 - \frac{2p - 2p^2}{p(a)} > 1 - \frac{2p - 2p^2}{1/2} = (2p - 1)^2 > 0.$$

The second case, for a even and at least 4, gives

$$1 - \frac{2p - 2p^2}{p(a)} > 1 - \frac{2p - 2p^2}{p(2)} = 0.$$

Now one can apply Chernoff bounds with the given ε to show that

$$\sum_{3 \leq a < \sqrt{n}} \binom{n}{a} \Pr(Bin(n-a, p(a)) \leq (2p - 2p^2)n - a) = o(1).$$

Now we bound the second sum corresponding to $\sqrt{n} < a \leq (2p - 2p^2)n$. Note that

$$\begin{aligned} & \sum_{\sqrt{n} \leq a \leq (2p-2p^2)n} \binom{n}{a} \Pr(Bin(n-a, p(a)) \leq (2p - 2p^2)n - a) \\ &= \sum_{\sqrt{n} \leq a \leq (2p-2p^2)n} \binom{n}{a} \Pr\left(Bin\left(n-a, \frac{1}{2} + O(e^{-\Omega(n^{1/2})})\right) \leq (2p - 2p^2)n - a\right) \\ &\leq n2^{nh(2p-2p^2)+o(1)} = o(1) \end{aligned}$$

since $p > p_0$ implies that $2p - 2p^2 < \lambda_0$.

3.2.2 $p = 1 - o(1)$

One can check it by following the same strategy as above and in Section 3.1.2.

3.3 $\lambda_0 \leq p \leq p_0$

Let $\alpha = 2\lambda_0/3$, $a = \lfloor \alpha n \rfloor$. Fix an a -set $A \subset V$ and generate our random graph and determine $B = B(A)$ with $b = |B|$. Let $\varepsilon = (\log n)^4/\sqrt{n}$ and let X_A be the indicator random variable for $a + b \leq (\lambda_0 + \varepsilon)n$ and $X = \sum_A X_A$. Then

$$p(a) = \frac{1}{2} + e^{-\Omega(n)}$$

and with $g_\lambda(\alpha)$ as defined in (4),

$$E(X) = \exp\{(g_{\lambda_0+\varepsilon}(2\lambda_0/3) + o(1))n \log 2\}. \quad (13)$$

Now

$$\begin{aligned} g_{\lambda+\varepsilon}(\alpha) &= g_\lambda(\alpha) + (1 - \alpha) \left(H\left(\frac{\lambda + \varepsilon - \alpha}{1 - \alpha}\right) - H\left(\frac{\lambda - \alpha}{1 - \alpha}\right) \right) \\ &= g_\lambda(\alpha) + \varepsilon \log_2 \left(\frac{1 - \lambda}{\lambda - \alpha} \right) + O(\varepsilon^2). \end{aligned}$$

Plugging this into (13) with $\lambda = \lambda_0$ and $\alpha = 2\lambda_0/3$ we see that

$$E(X) = \exp \left\{ \left(\varepsilon \log_2 \left(\frac{1 - \lambda_0}{\lambda_0/3} \right) + O(\varepsilon^2) \right) n \log 2 \right\} = e^{\Omega((\log n)^4 n^{1/2})}. \quad (14)$$

Next, we estimate the variance of X . We will argue that for $A, C \in \binom{V}{a}$ either $|A \triangle C|$ is small (but the number of such pairs is small) or $|A \triangle C|$ is large (but then the covariance $Cov(X_A, X_C)$ is very small since if we fix the adjacency of some vertex x to C , then the parity of $|N(x) \cap (A \setminus C)|$ is almost a fair coin flip). Formally,

$$\begin{aligned} Var(X) &= E(X) + \sum_{A \neq C} Cov(X_A, X_C) \\ &\leq E(X) + \sum_{|A \triangle C| < 2\sqrt{n}} \Pr(X_A = X_C = 1) \\ &\quad + \sum_{|A \triangle C| \geq 2\sqrt{n}, |A \cap C| \geq \sqrt{n}} Cov(X_A, X_C) \\ &\quad + \sum_{|A \cap C| < \sqrt{n}} \Pr(X_A = X_C = 1). \end{aligned}$$

Since $E(X)$ goes to infinity, clearly $E(X) = o(E(X)^2)$. We show in Claims 3.1, 3.2 and 3.3 that the remaining part is also bounded by $o(E(X)^2)$. Then Chebyshev's inequality will imply that $X > 0$ asymptotically almost surely.

Claim 3.1 $\sum_{|A \triangle C| < 2\sqrt{n}} \Pr(X_A = X_C = 1) = o(E(X)^2)$

Proof. We estimate trivially $\Pr(X_A = X_C = 1) \leq \Pr(X_A = 1)$. Then,

$$\begin{aligned} \sum_{|A \Delta C| < 2\sqrt{n}} \Pr(X_A = 1) &= \binom{n}{a} \sum_{0 \leq i < \sqrt{n}} \binom{n-a}{i} \binom{a}{a-i} \Pr(X_A = 1) \\ &= E(X) \sum_{0 \leq i < \sqrt{n}} \binom{n-a}{i} \binom{a}{a-i} \\ &\leq E(X) 2^{O(\sqrt{n} \log n)}. \end{aligned}$$

Thus, (14) yields that $\sum_{|A \Delta C| < 2\sqrt{n}} \Pr(X_A = X_C = 1) = o(E(X)^2)$. \square

Claim 3.2 $\sum_{|A \Delta C| \geq 2\sqrt{n}, |A \cap C| \geq \sqrt{n}} \text{Cov}(X_A, X_C) = o(E(X)^2)$

Proof. If $x \in V \setminus (A \cup C)$, then $\Pr(x \in B(A) | X_C = 1) = 2^{-1+o(1/n)}$, since we can always find at least \sqrt{n} vertices in $A \setminus C$ with no adjacency with x determined by the event $X_C = 1$. Similarly, if $x \in C \setminus A$, then there are at least $\sqrt{n} - 1$ vertices in $A \cap C$ such that their adjacency with x is independent of the occurrence of $X_C = 1$. This implies that

$$\Pr(X_A = 1 | X_C = 1) = \sum_{0 \leq i \leq l-a} \binom{n-a}{i} 2^{-(n-a)+o(1)} = 2^{o(1)} \Pr(X_A = 1),$$

and consequently, $\text{Cov}(X_A, X_C) = o(\Pr(X_A = 1)^2)$. Hence,

$$\sum_{|A \Delta C| \geq 2\sqrt{n}, |A \cap C| \geq \sqrt{n}} \text{Cov}(X_A, X_C) \leq \binom{n}{a}^2 o(\Pr(X_A = 1)^2) = o(E(X)^2).$$

\square

Claim 3.3 $\sum_{|A \cap C| < \sqrt{n}} \Pr(X_A = X_C = 1) = o(E(X)^2)$

Proof. First let us estimate the number of ordered pairs (A, C) for which $|A \cap C| < \sqrt{n}$. Note,

$$\begin{aligned} \sum_{|A \cap C| < \sqrt{n}} 1 &= \binom{n}{a} \sum_{0 \leq i < \sqrt{n}} \binom{n-a}{a-i} \binom{a}{i} \\ &\leq \sqrt{n} \binom{n}{a} \binom{n-a}{a} \binom{a}{\sqrt{n}} \\ &= 2^{n(H(\alpha) + H(\frac{\alpha}{1-\alpha})(1-\alpha) + o(1))}. \end{aligned} \tag{15}$$

Now we will bound $\Pr(X_A = X_C = 1)$ for fixed a -sets A and C . Let $S \subset A \setminus C$ be a set of size $s = |S| = \lfloor \sqrt{n} \rfloor$. Define a new indicator random variable Y

which takes the value 1 if and only if $|B(C) \setminus S| \leq (\lambda_0 + \varepsilon)n - a$. Clearly, $X_C \leq Y$ and

$$\begin{aligned} \Pr(Y = 1) &= \Pr(Bin(n - a - s, p(a)) \leq (\lambda_0 + \varepsilon)n - a) \\ &\leq 2^{s+o(1)} \sum_{0 \leq i \leq (\lambda_0 + \varepsilon)n - a} \binom{n-a}{i} 2^{-(n-a)} \\ &= 2^{s+o(1)} \Pr(X_A = 1), \end{aligned}$$

Now if we condition on the existence or otherwise of all edges F' between C and $V \setminus S$ then if $x \in V \setminus A$

$$\Pr(x \in B(A) \mid F' \text{ and } F'') \in \left[\frac{1 - (1 - 2p)^s}{2}, \frac{1 + (1 - 2p)^s}{2} \right],$$

where F'' is the set of edges between x and $A \setminus S$. This implies that

$$\begin{aligned} \Pr(X_A = 1 \mid Y = 1) &= \sum_{0 \leq i \leq (\lambda_0 + \varepsilon)n - a} \binom{n-a}{i} 2^{-(n-a)+o(1)} \\ &= 2^{o(1)} \Pr(X_A = 1), \end{aligned}$$

Consequently,

$$\Pr(X_A = X_C = 1) \leq \Pr(X_A = Y = 1) \leq 2^{\sqrt{n}+o(1)} \Pr(X_A = 1)^2.$$

Hence, (15) implies

$$\sum_{|A \cap C| < \sqrt{n}} \Pr(X_A = X_C = 1) \leq 2^{n(H(\alpha) + H(\frac{\alpha}{1-\alpha})(1-\alpha) + o(1))} \Pr(X_A = 1)^2.$$

To complete the proof it is enough to note that

$$E(X)^2 = 2^{n(2H(\alpha) + o(1))} \Pr(X_A = 1)^2$$

and

$$2H(\alpha) > H(\alpha) + H\left(\frac{\alpha}{1-\alpha}\right)(1-\alpha).$$

Indeed, the last inequality follows from the strict concavity of the entropy function, since then $(1-\alpha)H\left(\frac{\alpha}{1-\alpha}\right) + \alpha H(0) \leq H(\alpha)$ with the equality for $\alpha = 0$ only. \square

Now we show that $f(G_{n,p}) \geq (\lambda_0 - \varepsilon)n$. We show that

$$\sum_{1 \leq a \leq (\lambda_0 - \varepsilon)n} \binom{n}{a} \Pr(Bin(n - a, p(a)) \leq (\lambda_0 - \varepsilon)n - a) = o(1).$$

As in previous sections we split this sum into two sums but this time we make the break into $1 \leq a \leq (\log n)^2$ and $(\log n)^2 < a \leq (\lambda_0 - \varepsilon)n$, respectively. In order to estimate the first sum we use the Chernoff bounds with deviation $1 - \theta$ from the mean where

$$\theta = 1 - \frac{(\lambda_0 - \varepsilon)n - a}{(n - a)p(a)} \geq 1 - \frac{\lambda_0 - \varepsilon}{p(a)} \geq 1 - \frac{\lambda_0 - \varepsilon}{\lambda_0} = \frac{\varepsilon}{\lambda_0}.$$

Consequently,

$$\begin{aligned} \sum_{2 \leq a < (\log n)^2} \binom{n}{a} \Pr(Bin(n - a, p(a)) \leq (\lambda_0 - \varepsilon)n - a) \\ \leq (\log n)^2 \binom{n}{(\log n)^2} \exp\{-\Omega(\log n)^4\} \\ \leq \exp\{-\Omega(-(\log n)^4)\} = o(1). \end{aligned}$$

Now we bound the second sum corresponding to $\log \log n < a \leq (\lambda_0 - \varepsilon)n$.

$$\begin{aligned} \sum_{\log \log n \leq a \leq (\lambda_0 - \varepsilon)n} \binom{n}{a} \Pr(Bin(n - a, p(a)) \leq (\lambda_0 - \varepsilon)n - a) \\ = 2^{n(h(\lambda_0 - \varepsilon) + o(1/n))} = o(1). \end{aligned}$$

4 General Graphs

Here we present the proof of Theorem 1.2. First, we prove a weaker result $f(n) \leq (0.440 \dots + o(1))n$.

Suppose we aim at showing that $f(n) \leq \lambda n$. We fix some α and ρ and let $a = \alpha n$ and $r = \rho n$. For each a -set A let $R(A)$ consist of all sets that have Hamming distance at most r from $B(A)$. If

$$\binom{n}{a} \sum_{i=0}^r \binom{n}{i} = 2^{n(H(\alpha) + H(\rho) + o(1))} > 2^n, \quad (16)$$

then there are A, A' such that $R(A) \cap R(A') \ni C$ is non-empty. This means that C is within Hamming distance r from both $B = B(A)$ and $B' = B(A')$. Thus $|B \triangle B'| \leq 2r$.

Let all vertices in $A'' = A \triangle A'$ flip their neighbors, *i.e.*, execute operation O_1 . The only vertices outside of A'' that can have an odd number of neighbors in A'' are restricted to $(B \triangle B') \cup (A \cap A')$. Thus

$$f(G) \leq |A \triangle A'| + |(B \triangle B') \cup (A \cap A')| \leq 2a + 2r = 2n(\alpha + \rho). \quad (17)$$

Consequently, we try to minimise $\alpha + \rho$ subject to $H(\alpha) + H(\rho) > 1$. Since the entropy function is strictly concave, the optimum satisfies $\alpha = \rho$, otherwise replacing each of α, ρ by $(\alpha + \rho)/2$ we strictly increase $H(\alpha) + H(\rho)$ without changing the sum. Hence, the optimum choice is

$$\alpha = \rho \approx 0.11002786443835959 \dots$$

the smaller root of $H(x) = 1/2$, proving that $f(n) \leq (0.440 \dots + o(1))n$.

In order to obtain a better constant we modify the approach taken in (16). Let us take $\delta = 0.275$, $\alpha = 0.0535$, $a = \lfloor \alpha n \rfloor$, $d = \lfloor \delta n \rfloor$. Look at the collection of sets $B(A)$, $A \in \binom{[n]}{a}$. This gives $\binom{n}{a} = 2^{n(H(\alpha) + o(1))}$ binary n -vectors.

We claim that some two of these vectors are at distance at most d . If not, then inequality (5.4.1) in [3] says that

$$H(\alpha) + o(1) \leq \min\{1 + g(u^2) - g(u^2 + 2\delta u + 2\delta) : 0 \leq u \leq 1 - 2\delta\},$$

where $g(x) = H((1 - \sqrt{1 - x})/2)$. In particular, if we take $u = 1 - 2\delta = 0.45$, we get $0.30108 + o(1) \leq 0.30103$, a contradiction.

Thus, we can find two different a -sets A and A' such that $|B(A) \triangle B(A')| \leq d$. As in (17), we can conclude that $f(G) \leq 2a + d \leq (0.382 + o(1))n$.

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